Zoom Class 29/04/2020

Asymptotes

Defn : A straight line at a finite distance from the origin to which a tangent to a curve tends as the point of contact tends to infinity, is called an asymptote of the curve.

If $y = mx + c$ is an asymptote to a curve then, $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$ and $c = \lim_{x\to\infty}(y - mx)$

And if $P(x,y)$ is any point on the curve and 'p' is the distance between the curve and asymptote then

$$p = \frac{y - mx - c}{\sqrt{1+m^2}}$$

Ques Find the asymptote of the curve $x^3 + y^3 = 3axy$.

Soln Let $y = mx + c$ is the asymptote. and given curve is

$$x^3 + y^3 = 3axy \quad \text{——— (1)}$$

$$\Rightarrow 1 + \left(\frac{y}{x}\right)^3 = 3a\frac{xy}{x^2} = 3a\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

Taking limit $x \to \infty$ we get

$$1 + \lim_{x\to\infty}\left(\frac{y}{x}\right)^3 = 3a\cdot\lim_{x\to\infty}\left(\frac{y}{x}\right)\cdot\lim_{x\to\infty}\left(\frac{1}{x}\right)$$

$\Rightarrow 1+m^3 = 3am \cdot 0$

$\Rightarrow 1+m^3 = 0$

$\Rightarrow (1+m)(m^2-m+1) = 0$

$\Rightarrow (1+m) = 0 \Rightarrow m = -1$

Now $c = \lim_{x\to\infty} (y-mx)$

$= \lim_{x\to\infty} (y+x)$

$\Rightarrow y = c-x. \quad — ②$

Substituting ② in ① we get

$x^3 + (c-x)^3 = 3ax(c-x)$

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$\Rightarrow \cancel{x^3} + c^3 - \cancel{x^3} + 3cx^2 - 3c^2x$
$= 3acx - 3ax^2$

$\Rightarrow c^3 + 3cx^2 - 3cx^2 - 3acx + 3ax^2 = 0$

$\Rightarrow (3c+3a)x^2 - 3cx(c+a) + c^3 = 0$

$\Rightarrow 3(c+a)x^2 - 3cx(c+a) + c^3 = 0$

~~$\Rightarrow (c+a)(3x^2-3cx)+c^3=0$~~

$\Rightarrow 3(c+a) - \frac{3c}{x}(c+a) + \left(\frac{c^3}{x^2}\right) = 0$

Taking limit $x\to\infty$ we get

$\Rightarrow 3(c+a) - 0 + 0 = 0$

$\Rightarrow 3(c+a) = 0 \Rightarrow c+a = 0 \Rightarrow c = -a$

$\therefore y = mx + c$

$\Rightarrow y = -x - a \Rightarrow \boxed{y+x+a=0}$

is required Equation.

Working Rule to find Asymptotes on Algebraic Curve

If $y = mx + c$ then we equate $\phi_n(m) = 0$ and find values of m i.e., $m = m_1, m_2, m_3, \ldots$

and $c = \dfrac{-\phi_{n-1}(m)}{\phi_n'(m)}$

Now if the equation is homogeneous first we convert it to the form $x^n \phi\left(\frac{y}{x}\right)$ and find the highest degree 'n'

then, substituting $x = 1$ and $y = m$ in highest degree terms we equate

$$\phi_n(m) = 0 .$$

and solve to get ~~m=1,2,3,~~ $m = m_1, m_2, m_3, \ldots$

Then we get $\phi_{n-1}(m)$ by putting $x = 1$ and $y = 0$ in second highest degree

And diffrentiate $\phi_n(m)$ to obtain $\phi_n'(m)$

Putting these values in $c = \dfrac{-\phi_{n-1}(m)}{\phi_n'(m)}$

we obtain c.

If for certain values of m

c is of the form $\frac{0}{0}$

We find 'c' by following formula i.e.,

$$\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-1}(m) = 0.$$

Lastly after obtaining all values of $m = m_1, m_2, \ldots$ and corresponding values of $c = c_1, c_2, \ldots$ we get the asymptotes

$$y = m_1 x + c_1$$
$$y = m_2 x + c_2$$
$$\ldots$$
$$\ldots \text{ and so on.}$$

Ques Find real asymptotes of the curve $x^3 + y^3 = 3axy$.

Soln Given curve is $x^3 + y^3 = 3axy$.

$$\Rightarrow x^3 + y^3 - 3axy = 0$$

$$\Rightarrow 1 + \left(\frac{y}{x}\right)^3 - 3a\left(\frac{y}{x}\right)\cdot\frac{1}{x} = 0.$$

$\therefore$ Highest degree is $n = 3$.

$$\therefore \phi_3(m) = 1 + m^3 = 0$$
$$\Rightarrow (1+m)(m^2 - m + 1) = 0$$
$$\Rightarrow m = -1 \quad \text{or} \quad m = \frac{1 \pm i\sqrt{3}}{2}$$

$\because$ we have to find real asymptotes $m = \frac{1 \pm i\sqrt{3}}{2}$ is not considered

Again $\phi_2(m) = -3am \cdot 1$

$\Rightarrow \phi_2(m) = -3am$

$\phi_3'(m) = 3m^2$

$\therefore c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = \cancel{+}\dfrac{(+\cancel{3}am)}{\cancel{3}m^2} = \dfrac{a}{m}$.

$\because m = -1 \Rightarrow c = \dfrac{a}{(-1)} = -a$.

$\therefore$ Required asymptote is

$y = mx + c$

$\Rightarrow y = -x - a$

$\Rightarrow \boxed{y + x + a = 0}$

Ans